A New Entropy for Hypergraphs

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Motivation

- Representation of structured information as hypergraphs.
- Entropy measures.
- Fine grained analysis of the structure and complexity of hypergraphs.
- Entropy vector: entropy values of all partial hypergraphs.

Notations and background

- Hypergraph \( H = (V, E = \{e_i, i = 1 \ldots m, e_i \subseteq V\}), |V| = n, |E| = m \).
- Incidence matrix \( I, L(H) = I(H) \lambda = (\langle e_i \cap e_j \rangle)_{i,j \in \{1 \ldots m\}} \).
- Normalized eigenvalues of \( L(H): \mu_i, i = 1 \ldots m \).
- Entropy \( S(H) = -\sum_{i=1}^{m} \mu_i \log_2(\mu_i) \).
- Partial hypergraph \( H' = (V', \{e_{i'} \in J\}), J \subseteq \{1 \ldots m\}, \cup_{i \in J} e_i \subseteq V' \subseteq V \) (here \( V'' = V \)). Notation: \( H' \leq H \).

Main definition: entropy vector

For \( i \leq m \):

\[
SE_i(H) = \{ S(H_i) \mid H_i = (V, E_i), H_i \leq H, |E_i| = i \}
\]

- set of entropy values of all partial hypergraphs of \( H \) whose set of hyperedges has cardinality \( i \), arranged in increasing order.

Entropy vector of the hypergraph \( H \):

\[
SE(H) = (SE_1(H), SE_2(H), \ldots SE_m(H))
\]

with \( 2^m - 1 \) coordinates.

A simple example

\[
\begin{align*}
& e_1 & e_2 & e_3 \\
& 1 & 2 & 3
\end{align*}
\]

- \( SE_1 \): three partial hypergraphs containing one hyperedge \( (e_1, e_2 \text{ and } e_3, \text{ respectively}) \). \( SE_1 = (0, 0, 0) \)
- \( SE_2 \): three partial hypergraphs containing two hyperedges.
  \( (e_1, e_2): L = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \), eigenvalues = 2 and 3, \( s_1 = -\frac{3}{2} \log_2 \frac{2}{3} \).
  \( (e_1, e_3): \text{ same reasoning.} \)
  \( (e_2, e_3): L = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \), eigenvalues = 1 and 3, \( s_2 = -\frac{3}{4} \log_2 \frac{4}{3} \).
  \( SE_2 = (s_2, s_1) \approx (0.81, 0.97, 0.97) \)
- \( SE_3 \): one partial hypergraph containing three hyperedges, i.e. \( H \).
  \( L = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \), eigenvalues = 1, 3 and 3, \( s_3 = -\frac{3}{2} \log_2 \frac{2}{3} \).
  \( SE_3 = (s_3) \approx (1.45) \)
- Entropy vector:
  \( SE(H) = (0, 0, 0, s_2, s_1, s_3) \approx (0, 0, 0.81, 0.97, 0.97, 1.45) \)

Some properties

- \( S(H) = 0 \) if and only if \( |E| = 1 \).
- \( S(H) = \log_2(n) - \log_2(|r(H)|) = \log_2(\mu) \), where \( r(H) = \frac{\text{rank}}{\text{det}} \).
- for and only if \( H \) is uniform (i.e. \( \forall e \in E, |e| = |r(H)| \) and the intersection of any two distinct hyperedges is empty (i.e. for all \( e, e' \) in \( E \) such that \( e \neq e', |e \cap e'| = 0 \)).
- Two isomorphic hypergraphs have the same entropy vectors.
- Lattice structures:
  - on \( \mathcal{H} \) (isomorphism classes of hypergraphs) for the partial ordering defined by the subhypergraph relation \( \leq \).
  - on \( SE_H = \{ SE(H) \mid H \in \mathcal{H} \} \) for Pareto partial ordering on vectors.
  \( H' \leq H \Rightarrow SE(H') \leq SE(H) \).

On going work

- Reducing the complexity \( (SE(H)) = 2^m - 1 \)
  - by discarding two small or two large partial hypergraphs;
  - by approximating the computation of entropy;
  - by considering only the leading principal matrices \( (m - 1) \) instead of \( 2^m - 1 \) after sorting the hyperedges by increasing cardinality.
- Relation between entropy and Zeta function:
  \[
  \zeta_H(s) = \text{Tr}(L(H)^{-s}) = \sum_{i=1}^{m} \mu_i^{-s}.
  \]
  where \( L(H) = \frac{L(H)}{\text{Tr}(L(H))} \).
  First results:
  \[
  \zeta_H(-1) = \ln(2) S(H), \ z'_H(0) = -\ln(\det(L(H))), \ \zeta_H(-s) = e^{s} R_s(H).
  \]
  where \( R_s(H) = \frac{1}{s} \ln(\sum_{i=1}^{m} \mu_i^s) \) (Renyi entropy).
- Illustrations and examples.

A few illustrations

- \( H \)
  \[
  SE(H) \approx (0, 0, 0.42, 0.50, 0.85, 0.83) \quad (0, 0, 0.63, 0.74, 0.75, 1.11).
  \]

References


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